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Cost accounting methods and periodic-review policies for serial inventory systems

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ABSTRACT

For reasons of mathematical tractability and historic convention, previous studies on periodic-review inventory control policies under uncertainty have typically accounted inventory related costs at the end of each review period. Inventory holding and shortage costs in reality, however, often accrue continuously in time. Given this discrepancy, it is necessary to understand the impact of end-of-period cost accounting. We address this issue for serial inventory systems adopting fixed-interval ordering in this paper. Our contribution is two-fold. First, we develop a model to evaluate and optimize a serial inventory system where inventory holding and shortage costs accrue continuously in time. This model includes discrete-time cost accounting that evaluates inventory holding and shortage costs at a single or multiple discrete points in a reorder interval as a special case. Second, we assess the effect of applying discrete-time cost accounting when inventory holding and shortage costs actually accrue continuously in time. We make three observations. First, for single-stage systems, end-of-period cost accounting generally results in very significant cost inefficiency and this cost inefficiency is bounded below by the ratio of inventory holding cost to backordering cost asymptotically as the reorder interval increases. Second, for multiple-stage systems, the cost inefficiency of end-of-period cost accounting decreases with the number of stages but is still significant for a typical serial system. Finally, extending cost accounting from the end to multiple discrete points in a reorder interval does not provide a significant modelling and computational advantage as compared to our continuous-time cost accounting model.

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1. Introduction

Studies on periodic-review inventory control policies under uncertainty began with the classic newsvendor model (Arrow et al., 1951; Bellman et al., 1955; Clark and Scarf, 1960). Because of the daily nature of newspapers, the newsvendor model accounted inventory related costs based on the inventory level at the end of the day. Following this tradition and for reasons of mathematical tractability, subsequent studies on periodic-review inventory control policies have typically accounted inventory related costs based on the inventory level at the end of each review period (see, for example, Chao and Zhou, 2009; Chen and Zheng, 1994; van Houtum et al., 2007, and Shang and Zhou, 2010). However, costs for holding inventory such as costs of capital tied up with stock, costs for storage space and facility, and costs due to spoilage and obsolescences, etc., usually accrue continuously in time. According to Nahmias (2009), the most significant cost component for holding inventory is the opportunity cost for the capital invested in inventory which accrues continuously in time. Similar arguments can also be made for the costs of backlogging shortage. In reality, the cost of stock-out depends on not only the amount but also the duration. Since the inventory level at the end of a review period ignores variations during the period, end-of-period cost accounting does not provide an accurate evaluation of these inventory related costs (Rudi et al., 2009). As a result, an inventory control policy that minimizes system cost evaluated based on the end-of-period inventory level may not be optimal. This calls for an evaluation of end-of-period cost accounting when inventory related costs actually accrue continuously in time.

We address this issue for serial inventory systems in this paper. Rudi et al. (2009) studied this issue for the classical single-period newsvendor model. They showed that, when inventory related costs actually accrue continuously in time, end-of-period cost accounting using the same continuous-time cost parameters leads to higher (than optimal) order-up-to inventory levels and hence higher inventory related costs. Nevertheless, they found that the use of end-of-period cost accounting can be justified by adjusting the cost parameters and demand distribution. Unfortunately, this

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approach may not be applicable for a serial inventory system. To the best of our knowledge, no method has been developed to evaluate and optimize a general serial inventory system where inventory holding and shortage costs accrue continuously in time. Consequently, even if there are adjustments on cost parameters and demand distribution that can justify the use of end-of-period cost accounting, it is difficult to identify such adjustments. As such, the objective of the current study is to first develop an approach to evaluate and optimize a serial inventory system where inventory related costs accrue continuously in time. Using this solution, we subsequently assess the impact of cost accounting methods on system stock and cost.

We consider an N-stage serial inventory system, in which customer demand arises at only the most downstream Stage 1 and Stage j orders exclusively from the immediate upstream Stage $j+1$ for $j=1,\dots,N$. Stage $N+1$ is assumed to be an external supplier that has unlimited capacity and hence is always able to meet an order from stage N immediately in its entirety. An internal stage j for $j=1,\dots,N$, however, can meet the demand from the next downstream stage (or from external customers for Stage 1) only when stock is available. Demand that is not satisfied by on-hand inventory is fully backlogged. Ordered stock is delivered after a constant leadtime.

In view of the literature, we assume that the system adopts fixed-interval ordering, in which each stage orders from the upstream stage in a fixed-interval. Fixed-interval ordering has become a common practice for supply chains to facilitate freight consolidations and logistics/production scheduling (Graves, 1996; Marklund, 2011). According to Chao and Zhou (2009), an optimal fixed-interval ordering policy for a serial inventory system must be nested and synchronized. This requires that Stage j orders in a fixed interval that is an integer multiple of the fixed interval at the downstream Stage $j-1$ and always synchronizes the arrival of a shipment from Stage $j+1$ with an order from Stage $j-1$ for $j=2,\dots,N$. Moreover, given a nested and synchronized replenishment schedule, previous studies have shown that echelon base-stock policies are optimal for both finite and infinite planning horizons (Chao and Zhou, 2009; Chen and Zheng, 1994; Clark and Scarf, 1960; Federgruen and Zipkin, 1984; van Houtum et al., 2007). Based on these findings, we consider a nested and synchronized fixed-interval echelon base-stock policy for the serial inventory system. To focus on the effects of cost accounting method, we assume that the system replenishment schedule is exogenously determined and therefore consider only inventory holding and backordering costs. The system cost structure includes a linear cost for holding one unit of stock per unit time at each stage and a linear cost for backlogging a unit of demand per unit time at Stage 1.

Periodic-review inventory control policies for multi-echelon inventory systems have been commonly evaluated by the echelon-stock approach. For an N-stage serial inventory system, echelon j refers to the portion of the system from the most downstream Stage 1 up to Stage j and echelon inventory level j (or alternatively echelon stock j) is defined as all inventories on hand or in transit in Echelon j minus backorders at Stage 1 for $j=1,\dots,N$. By this definition, the difference between echelon inventory levels j and $j-1$ represents the inventory on hand at Stage j plus the inventory in transit from Stage j to Stage $j-1$. Consequently, at a given point in time, the expected inventory at Stage j for $j=1,\dots N$ and backorder at Stage 1 can be evaluated by characterizing the (limiting) distributions for the echelon inventory levels $j=1,\dots N$. The instantaneous expected inventory holding and backordering costs can be obtained accordingly by multiplying the expectations by the appropriate cost parameters. Under end-of-period cost accounting, expected inventory related costs are evaluated at the end of the fixed reorder intervals at Stage 1. This cost evaluation method began in the seminal papers by Arrow et al. (1951) and Bellman et al. (1955) and has become a convention for studies on periodic-review inventory models (Chen and Zheng, 1994; Clark and Scarf, 1960; Chao and Zhou, 2009; van Houtum et al., 2007; Shang and Zhou, 2010).

Technically, the echelon-stock method can be used under any cost accounting scheme provided that the (limiting) distributions for all echelon inventory levels can be characterized and the instantaneous expected inventory related costs can be integrated accordingly over a system reorder cycle. These tasks, however, seem to be cumbersome when inventory holding and backordering costs accrue continuously in time. In view of the literature, Rao (2003) started the recent interest in periodic-review inventory control policies under continuous-time cost accounting. The objective of his study was to compare the commonly used periodic-review (R, T) policy, in which stock is ordered in a fixed interval T to raise the inventory position to R, to the continuous-review (Q, r) policy, in which a fixed batch of size Q is ordered whenever the inventory position drops to r, for a single-stage system. Because the (Q, r) policy monitors the inventory position and evaluates the inventory related costs continuously in time, the (R, T) policy must be evaluated and optimized accordingly. His study highlighted the reality whereby inventory related costs often accrue continuously in time and led to other studies on periodic-review inventory control policies under continuous-time and other cost accounting schemes (Avinadav, 2015; Avinadav and Henig, 2015; Liu and Song, 2012; Rudi et al., 2009). Subsequently, Feng and Rao (2007) attempted to extend the (R, T) policy to a two-stage serial system under continuous-time cost accounting. Due to difficulties in finding a tractable evaluation, they used simulation to evaluate the system cost. More recently, Wang (2013) considered a two-level distribution system that consists of a warehouse and a group of identical retailers adopting a common fixed reorder interval. Assuming that warehouse stock is optimally allocated to minimize system cost when facing a shortage, he developed an approach to evaluate the inventory related costs at the retailers continuously in time based on the stocking conditions at the warehouse. Meanwhile, Wang and Liu (2015) revisited the two-stage serial system that was considered by Feng and Rao (2007). They observed that, given the time elapsed from a warehouse reorder point until the demand at the retailer accumulates to the warehouse (installation) stock level, the expected cost at the retailer can be evaluated as a single-stage (R, T) policy with an extension on the leadtime. They developed a simple cost evaluation for the system when demand follows a Poisson process. In sum, the literature on periodic-review inventory control policies for inventory systems where inventory related costs accrue continuously in time is still very limited. This is particularly the case for multi-echelon inventory control systems.

In this paper, we develop an approach to evaluate a general serial inventory system. In contrast to the echelon-stock approach, we develop our evaluation based on installation stock. Specifically, we start with Echelon 1 and observe that the expected cost in a reorder cycle is completely determined by the installation stock at Stage 2 at the time of ordering. As such, a cost evaluation can be developed conditioned on the installation stock at Stage 2. In the same way, the expected cost in a reorder cycle at Echelon 2 can be evaluated conditioned on the installation stock at Stage 3, etc. Using this observation, we develop a recursive approach to evaluate the expected inventory holding and backordering costs in the system. This approach uses only the demand distribution, is easy to follow and compute, and can be used under any cost accounting method.

Subsequently, we characterize the optimal order-up-to levels under a given replenishment schedule. Using the explicit cost evaluation that is developed in this study, we derive a simple way to determine the optimal base stock level for each stage or echelon.

Specifically, consider a unit of stock at Stage $j+1$ at a reorder point for the downstream Stage j. Intuitively, this unit of stock should be shipped to Stage j if the marginal cost at Echelon j is lower than the cost for holding it at Stage $j+1$. Consequently, an optimal base stock level at Stage j is reached when the two costs are equal. We prove that a unique optimal ordering policy is determined by this condition at each echelon given the base stock levels at the downstream stages. This leads to a simple bottom-up recursive approach to identify an optimal ordering policy for a given replenishment schedule.

Finally, we assess the effect of cost accounting methods. We start with single-stage systems and demonstrate numerically that end-of-period cost accounting generally results in extensive excess stock and hence very significant cost inefficiency. Moreover, we prove that both the excess stock and cost inefficiency are bounded below by the ratio of inventory holding cost to backordering cost asymptotically as the reorder interval increases for single-stage systems. Subsequently, we consider multiple-stage serial inventory systems and show numerically that the excess stock and cost inefficiency caused by end-of-period cost accounting decrease with the number of stages. Nevertheless, they are still often significant for typical three-stage serial inventory systems. Finally, in view of the literature, we consider extending end-of-period cost accounting to discrete-time cost accounting that evaluates system costs at multiple discrete points in each reorder interval of Stage 1. Our numerical studies show that, given a typical approximation accuracy of 2%, this extension normally does not provide any computational advantage as compared to the solution we develop under continuous-time cost accounting in the current study.

The rest of the paper is organized as follows. Section 2 formulates the model, Section 3 develops the system cost evaluation, Section 4 characterizes the optimal ordering policy, Section 5 presents a numerical study, and finally Section 6 concludes the study.

2. Model formulation

We consider an N-stage serial inventory system (see Fig. 1) and assumes that the system adopts a synchronized and nested echelon base-stock policy. Specifically, we let Stage j order from Stage $j+1$ in a fixed interval of length T_j to raise the echelon inventory position (abbreviated as IP hereafter) to a fixed base-stock level S_j, whereby $T_{j+1} = n_{j+1}T_j$ and n_{j+1} is a positive integer and Stage j places an order upon the arrival of every shipment at Stage $j+1$ for $j = 1, \ldots, N$. Stage $N+1$ is defined as an external supplier with unlimited stock and echelon IP is defined as inventory on hand or in transit in the echelon plus outstanding orders minus backorders at Stage 1. By this definition, the replenishment schedule and ordering policy at echelon $j = 1, \ldots, N$ are fully defined by $\mathbf{T}_j = (T_1, \ldots, T_j)$ and $\mathbf{S}_j = (S_1, \ldots, S_j)$, respectively. To facilitate the analysis below, we note that $\mathbf{S}_j$ can be defined equivalently based on installation stock. Namely, let $R_1 = S_1$ and $R_j = S_j - S_{j-1}$ for $j > 1$. The ordering policy at Echelon j is equivalent to raising the installation IP at Stage j to R_j and the echelon IP at Stage $j-1$ to S_{j-1}. We refer to $\mathbf{R}_j = (R_1, \ldots, R_j)$ as the installation-stock definition of the ordering policy at Echelon $j = 1, \ldots, N$ hereafter.

In addition, we define the following parameters for $j = 1, \ldots, N$:

λ = mean customer demand rate per unit time,

l_j = constant leadtime to ship stock from Stage $j+1$ to Stage j,

h_j = installation cost for holding one unit of stock per unit time at Stage j, and

b = cost for backlogging one unit of demand per unit time at Stage 1.

We assume that the demand at Stage 1 has stationary and independent increments. This holds true for the commonly used compound Poisson process and the normal demand model. Let $X(t) = D[a, a+t)$ denote the cumulative demand over the interval $[a, a+t)$ of length t. We assume that $X(t)$ is stochastically non-decreasing in t with mean λt, density or mass function $g(x, t)$, and cumulative distribution function (c.d.f.) $G(x, t)$ (Liu and Song, 2012; Rao, 2003). We conduct the analysis for the case of continuous demand, i.e., $G(x,t) = \int_0^x g(y,t)dy$. When demand is discrete, we need only to change the integration to summation accordingly, i.e., $G(x,t) = \sum_{y=0}^{x} g(y,t)$.

To facilitate the analysis below, we also define the following cost functions for Echelon $j = 1, \ldots, N$:

$\pi_j(R_j, T_j| \cdot)$ = the expected cumulative inventory holding and backordering cost in a replenishment cycle and

$C_j(R_j, T_j|\cdot) = \pi_j(R_j, T_j|\cdot)/T_j$ = the expected inventory holding and backordering cost per unit time in a replenishment cycle, given the replenishment schedule $\mathbf{T}_j$ and ordering policy $\mathbf{R}_j$.

We note that $\pi_j(R_j, T_j| \cdot)$ and $C_j(R_j, T_j| \cdot)$ are actually functions of $\mathbf{R}_j$ and $\mathbf{T}_j$. We keep only R_j and T_j in the notation for simplicity and convenience of analysis below.

3. Evaluation of expected inventory related costs in the system

Under the inventory control policy that is defined above, the system regenerates every time when Stage N orders from the external supplier. To evaluate the long-run system cost, we need to consider only one system regenerative cycle. The long-run average inventory related cost per unit time in the system is then given by the obtained expected cost divided by the cycle length.

We start with Stage 1. The following basic evaluation is straightforward (Wang, 2013).

Lemma 1. *Let Stage 1 order from Stage 2 at time 0 (normalized to time 0 if otherwise), raising the IP at Stage 1 to R_1. Then the expected inventory holding and backordering cost at Stage 1 in the interval $[l_1, l_1+T_1)$ is given by*

$$\pi_1(R_1, T_1) = b[\lambda(T_1/2 + l_1) - R_1]T_1 + (h_1 + b)I_1(R_1, T_1) \tag{1}$$

where $I_1(R_1, T_1) = \int_{l_1}^{l_1+T_1} \int_0^{R_1} G(x,t)dxdt$ is the expected on-hand inventory in the interval.

Proof. At any time $t \in [l_1, l_1+T_1)$, the inventory level at Stage 1 is equal to $R_1 - D[0,t)$ and hence the instantaneous expected inventory holding and backordering cost is equal to

$$\begin{aligned} u(R_1, t) &= E[h_1(R_1 - D[0,t))^+ + b(D[0,t) - R_1)^+] \\ &= (h_1 + b)\int_0^{R_1}(R_1 - x)g(x,t)dx + b(\lambda t - R_1) \end{aligned} \tag{2}$$

where $(x)^+ = \max\{0, x\}$. The cumulative expected inventory holding and backordering cost in the interval $[l_1, l_1+T_1)$ is then given by $\pi_1(R_1, T_1) = \int_{l_1}^{l_1+T_1} u(R_1, t)dt$, which yields (1). □

We note that $\pi_1(R_1, T_1|\cdot) = \pi_1(R_1, T_1)$ for Stage 1.

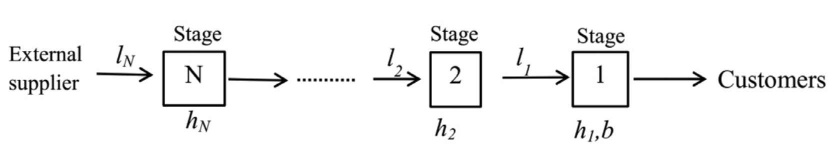

Fig. 1. N-stage serial system.

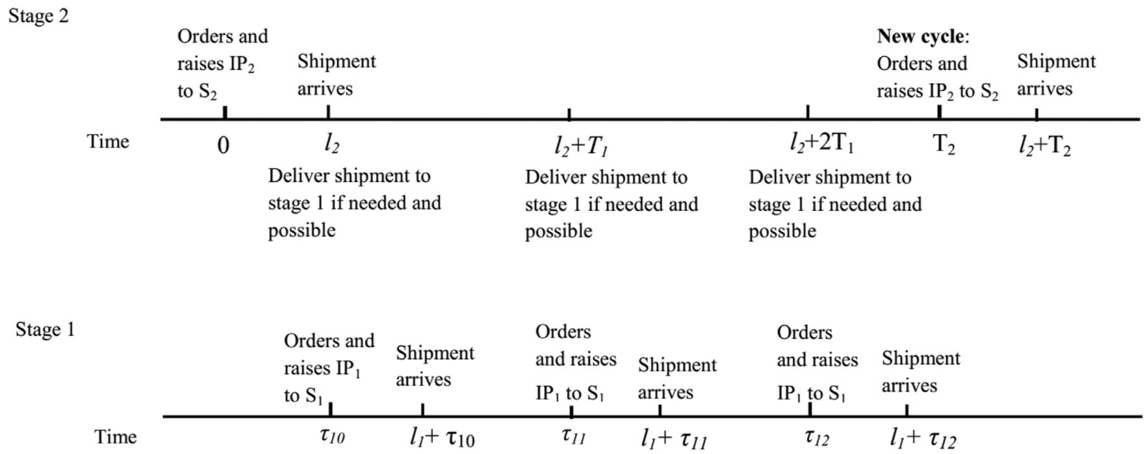

Fig. 2. A system regenerative cycle for a two-stage system.

We consider now Echelon $j = 2, \ldots, N$. Similar to the above, we consider one replenishment cycle of length T_j that starts at time 0 when Stage j orders from Stage $j+1$, raising the echelon IP to S_j. The shipment from Stage $j+1$ arrives at time l_j and Stage $j-1$ is scheduled to order from Stage j to raise the echelon IP to S_{j-1} at time $\tau_{jq} = l_j + qT_{j-1}$ for $q = 0, \ldots, n_j - 1$.

Fig. 2 illustrates a regenerative replenishment cycle for a two-stage serial system with $T_2 = 3T_1$. We let Stage 2 order at time 0, raising the echelon or system base-stock level to S_2. Stage 2 will then order again at time T_2, raising the system base-stock level to S_2. The system regenerates at time T_2. During this system replenishment cycle, the stock ordered by Stage 2 from the external supplier at time 0 will arrive after a lead time of l_2 and Stage 1 will order from Stage 2 to raise its base-stock level to $S_1 (= R_1)$ at time $\tau_{1i} = l_2 + iT_1$ for $i = 0, 1, 2$. We note that Stage 2 can make a delivery to Stage 1 at time τ_{1i} only if it does not run out of stock before this reorder point.

Because demand is uncertain, Stage j may run out of stock at time τ_{jq}. Should this happen, Stage j has no more stock to meet any order from Stage $j-1$ thereafter until receiving the next shipment from Stage $j+1$ at time $l_j + T_j$. To define the stocking conditions at Stage j, we let $Y_{jq-1} = D[0, \tau_{jq-1})$ and $X_{jq} = D[\tau_{jq-1}, \tau_{jq})$ denote the cumulative demand in the intervals $[0, \tau_{jq-1})$ and $[\tau_{jq-1}, \tau_{jq})$ respectively, for $q = 0, \ldots, n_j - 1$, where $\tau_{j0-1} \equiv 0$ and $Y_{j0-1} \equiv 0$. Since the order-up-to policy requires that Stage $j-1$ order the exact stock to replace the demand X_{jq} at time τ_{jq}, the total order quantity from Stage $j-1$ since time τ_{j0} until time τ_{jq} is equal to Y_{jq} and the remaining stock at Stage j at time τ_{jq} after meeting the order from Stage $j-1$ is equal to $R_j - Y_{jq}$ or zero, whichever is larger. This means that Stage j runs out of stock at time τ_{jq} if (i) $X_{jq} \geq R_j$ for $q = 0$, (ii) $Y_{jq-1} < R_j$ and $X_{jq} \geq R_j - Y_{jq-1}$ for $q = 1, \ldots, n_j - 1$ and (iii) $Y_{jq-1} < R_j$ for $q = n_j$. We note that these events are mutually exclusive and collectively exhaustive and the case for $q = n_j$ represents the event where Stage j does not run out of stock in the replenishment cycle.

We evaluate $\pi_j(R_j, T_j| \cdot)$ by the sum of the expected inventory holding cost at Stage j and the expected inventory holding and backordering cost at Echelon $j-1$ in the replenishment cycle.

Theorem 1. *Given $\pi_1(R_1, T_1)$, $\pi_j(R_j, T_j| \cdot)$ can be computed recursively starting from $j = 2$ by*

$$\pi_j(R_j, T_j|\cdot) = h_j T_j \int_0^{R_j} \Phi^j(y)dy + n_j \pi_{j-1}(R_{j-1}, T_{j-1}|\cdot)\Phi^j(R_j) + n_j \int_{R_j}^{\infty} \pi_{j-1}(R_{j-1} + R_j - y, T_{j-1}|\cdot)\phi^j(y)dy \quad (3)$$

where $\phi^j(y) = \sum_{q=0}^{n_j-1} g(y, \tau_{jq})/n_j$ and $\Phi^j(y) = \int_0^y \phi^j(x)dx$.

Proof. Following the discussions above, we assume that Stage j orders at time 0, raising the echelon IP to S_j, and evaluate the expected cost at Stage j in the interval $[l_j, l_j + T_j)$ and the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + l_j, l_{j-1} + l_j + T_j)$.

Since Stage j holds a stock that is equal to $R_j - Y_{jq}$ or zero, whichever is larger, in the interval $[\tau_{jq}, \tau_{jq+1})$ for $q = 0, \ldots, n_j - 1$, the cumulative expected inventory holding cost at Stage j in the interval $[l_j, l_j + T_j)$ is given by $h_j T_{j-1} \sum_{q=0}^{n_j-1} \int_0^{R_j} (R_j - y)g(y, \tau_{jq})dy = h_j T_j \int_0^{R_j} \Phi^j(y)dy$.

The remaining task is to evaluate the expected cost at Echelon $j-1$. We let $c_q^m(\cdot)$ denote the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + \tau_{jq}, l_{j-1} + \tau_{jq+1})$ given that Stage j runs out of stock at time τ_{jm} for $m = 0, \ldots, n_j$ and $q = 0, \ldots, n_j - 1$. By this definition, $\sum_{m=0}^{n_j} c_q^m(\cdot)$ is the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + \tau_{jq}, l_{j-1} + \tau_{jq+1})$ for $q = 0, \ldots, n_j - 1$ and $\sum_{q=0}^{n_j-1} \sum_{m=0}^{n_j} c_q^m(\cdot)$ is the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + l_j, l_{j-1} + l_j + T_j)$.

We break the summation $\sum_{m=0}^{n_j} c_q^m(\cdot)$ at $m = q$ and obtain

$$\sum_{q=0}^{n_j-1} \sum_{m=0}^{n_j} c_q^m(\cdot) = \sum_{q=0}^{n_j-1} \sum_{m=0}^{q} c_q^m(\cdot) + \sum_{q=0}^{n_j-1} \sum_{m=q+1}^{n_j} c_q^m(\cdot). \quad (4)$$

According to the definition above, $\sum_{m=q+1}^{n_j} c_q^m(\cdot)$ is the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + \tau_{jq}, l_{j-1} + \tau_{jq+1})$ given that Stage j runs out of stock at time τ_{jm} for $m = q+1, \ldots, n_j$, or equivalently after time τ_{jq}. This event happens if and only if $Y_{jq} < R_j$. Under this condition, Stage j has sufficient stock to raise the IP at Echelon $j-1$ to S_{j-1} at time τ_{jq} and the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + \tau_{jq}, l_{j-1} + \tau_{jq+1})$ is then equal to $\pi_{j-1}(R_{j-1}, T_{j-1}|\cdot)$. This leads to $\sum_{m=q+1}^{n_j} c_q^m(\cdot) = \int_0^{R_j} \pi_{j-1}(R_{j-1}, T_{j-1}|\cdot)g(y, \tau_{jq})dy = \pi_{j-1}(R_{j-1}, T_{j-1}|\cdot)G(R_j, \tau_{jq})$ since Y_{jq} is independent of the demand after time τ_{jq} and

$$\sum_{q=0}^{n_j-1} \sum_{m=q+1}^{n_j} c_q^m(\cdot) = \pi_{j-1}(R_{j-1}, T_{j-1}|\cdot) \sum_{q=0}^{n_j-1} G(R_j, \tau_{jq}). \quad (5)$$

On the other hand, $\sum_{m=0}^{q} c_{jq}^m(\cdot)$ is the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + \tau_{jq}, l_{j-1} + \tau_{jq+1})$ given that Stage j runs out of stock at time τ_{jm} for $m = 0, \ldots, q$, or equivalently before or at time τ_{jq}. This event happens if and only if $Y_{jq} \geq R_j$. Under this condition, all stock available at Stage j has been shipped to Stage $j-1$ by time τ_{jq}. The IP at Echelon $j-1$ at time τ_{jq} after receiving the shipment from Stage j (if any) is equal to $S_{j-1} + R_j - y$ given $Y_{jq} = y \geq R_j$. This yields $\sum_{m=0}^{q} c_q^m(\cdot) = \int_{R_j}^{\infty} \pi_{j-1}(R_j + R_{j-1} -$

$y, T_{j-1}|\cdot)g(y, \tau_{jq})dy$ and

$$\sum_{q=0}^{n_j-1}\sum_{m=0}^{q} c_q^m(\cdot) = \int_{R_j}^{\infty} \pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot) \sum_{q=0}^{n_j-1} g(y, \tau_{jq})dy. \tag{6}$$

Adding (5), (6) and the expected inventory holding cost at Stage j together yields (3). □

Alternatively, to facilitate computation, we can evaluate (6) by

$$\sum_{q=0}^{n_j-1}\sum_{m=0}^{q} c_q^m(\cdot) = \tilde{\pi}_{j-1}(R_j + R_{j-1}, T_j|\cdot) - \int_0^{R_j} \pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot) \sum_{q=0}^{n_j-1} g(y, \tau_{jq})dy \tag{7}$$

where $\tilde{\pi}_{j-1}(R_j + R_{j-1}, T_j|\cdot)$ is equal to $\pi_{j-1}(R_j + R_{j-1}, T_j|\cdot)$ with leadtime $l_{j-1} + l_j$.

We place the derivation of (7) in Appendix A for simplicity of presentation.

Assume that Stage $j+1$ is perfectly reliable. Then the expected inventory holding and backordering cost per unit time at Echelon j is equal to $C_j(R_j, T_j|\cdot) = \pi_j(R_j, T_j|\cdot)/T_j$ for $j = 1, \ldots, N$.

Theorem 1 provides a simple cost evaluation for a general serial inventory system that is easy to follow and compute. Several points about this cost evaluation are worth noting.

First, if demand is discrete, we need only to change the integrations with respect to demand into summations accordingly for the above evaluations.

Second, if costs are accounted discretely at m time points at time $l_1 \le t_i \le l_1 + T_1$ in the interval $[l_1, l_1 + T_1)$ for $i = 1, \ldots, m,$ we can evaluate the expected average inventory holding and backordering cost at Stage 1 in the interval $[l_1, l_1 + T_1)$ according to $\pi_1(R_1, T_1) = \sum_{i=1}^{m} u(t_i)/m,$ where $u(t_i)$ is given by (2). All other evaluations still hold.

Third, the expected inventory in-transit from Stage $j+1$ to Stage j is equal to λl_j per unit time. The cost for holding this expected inventory is constant and thus omitted.

Finally, $\pi_1(R_1, T_1) = b[\lambda(T_1/2 + l_1) - R_1]T_1 > \pi_1(0, T_1)$ for any $R_1 < 0$ and, if $R_j < 0$ for any $j > 0$, we can redefine the base stock level at Stage j to be zero and the base stock level at the downstream Stage $j-1$ to be $R_{j-1} + R_j$ without affecting the system cost. Consequently, we need to consider only $R_j \ge 0$ for $j = 1, \ldots, N$ in the optimization analysis below.

4. Optimal ordering policy

We discuss in this section the optimal ordering policy for a given replenishment schedule or the ordering policy $\mathbf{R_N}$ that minimizes $C_N(R_N, T_N| \cdot)$ given the replenishment schedule $\mathbf{T_N}$.

Lemma 2. *Assume $R_j > 0$ for $j = 1, \ldots N$. Then*

$$\frac{\partial C_N(R_N, T_N|\cdot)}{\partial R_j} - \frac{\partial C_N(R_N, T_N|\cdot)}{\partial R_{j+1}} = [\frac{\partial C_j(R_j, T_j|\cdot)}{\partial R_j} - h_{j+1}]P_j^N(R_N|\cdot) \tag{8}$$

for $j = 1, \ldots, N-1$ where $P_j^N(R_N|\cdot)$ is a function of $\mathbf{R_N}$.

Proof. For simplicity of presentation, we place the proof of Lemma 2 in Appendix B. □

The finding in Lemma 2 leads to a simple characterization of an optimal ordering policy.

Theorem 2. *For a given system replenishment schedule $\mathbf{T}_N$, an optimal ordering policy that minimizes $C_N(R_N, T_N| \cdot)$ is uniquely determined by the following bottom-up procedure.*

(i) Start with Stage 1. Identify R_1^* by $\partial C_1(R_1, T_1)/\partial R_1 = h_2$ and proceed to step (ii) below.

(ii) For Stage $j > 1$, given R_1^*, ..., R_{j-1}^* that are previously identified, identify the optimal R_j^* by $\partial C_j(R_j, T_j|\cdot)/\partial R_j = h_{j+1}$. If the obtained solution is $R_j^* = 0$, then Stage j is a virtual stage that does not carry any stock and Stage $j-1$ orders directly from Stage $j+1$. Revise R_{j-1}^* by $\partial C_{j-1}(R_{j-1}, T_{j-1}|\cdot)/\partial R_{j-1} = h_{j+1}$ with reorder interval T_j and lead time $l_j + l_{j-1}$. Repeat this step until Stage N with $h_{N+1} = 0$.

Proof. We consider the necessary conditions of optimality, i.e., $\partial C_N(R_N, T_N|\cdot)/\partial R_j = 0$ for $j = 1, \ldots, N$. Using Lemma 2, it is easy to verify that these necessary conditions are equivalent to $\partial C_j(R_j, T_j|\cdot)/\partial R_j = h_{j+1}$ for $j = 1, \ldots, N$ with $h_{N+1} = 0$. A solution of these conditions can be obtained using the bottom-up procedure in the theorem.

We prove below that this solution is unique and provides a minimum optimal solution. To this end, we obtain the second derivatives of $C_j(R_j, T_j| \cdot)$ w.r.t R_j and R_i for $i = 1, \ldots, j-1$:

$$\frac{\partial^2 C_j(R_j, T_j|\cdot)}{\partial R_j^2} = [h_j - \frac{\partial C_{j-1}(R_{j-1}, T_{j-1}|\cdot)}{\partial R_{j-1}}]\phi_j(R_j) + \int_{R_j}^{\infty} \frac{\partial^2 C_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)}{\partial R_{j-1}^2}\phi_j(y)dy, \tag{9}$$

$$\frac{\partial^2 C_j(R_j, T_j|\cdot)}{\partial R_j \partial R_i} = \sum_{q=0}^{n_j-1} \int_{R_j}^{\infty} \frac{\partial^2 C_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)}{\partial R_j \partial R_i}\phi_j(y)dy \text{ and} \tag{10}$$

$$\frac{\partial^2 C_j(R_j, T_j|\cdot)}{\partial R_i^2} = \frac{\partial^2 C_{j-1}(R_{j-1}, T_{j-1}|\cdot)}{\partial R_i^2}\Phi_j(R_j) + \int_{R_j}^{\infty} \frac{\partial^2 C_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)}{\partial R_i^2}\phi_j(y)dy. \tag{11}$$

It is easy to verify that $C_1(R_1, T_1)$ is convex in R_1 and R_1^* provides a unique minimum solution.

Given $R_1 = R_1^*$, we apply the conditions: $h_2 - \partial C_1(R_1, T_1)/\partial R_1 = 0$ and $\partial^2 C_1(R_1, T_1)/\partial R_1^2 \ge 0$ in (9), (10) and (11), respectively, for $j = 2$ and obtain $\partial^2 C_2(R_2, T_2| \cdot)/\partial R_i \partial R_q \ge 0$ for $i, q = 1, 2$. This means that $C_2(R_2, T_2| \cdot)$ is convex in R_2 and super-modular in R_1 and R_2. Therefore, if $R_2^* > 0$, R_2^* provides a unique minimum solution. If $R_2^* = 0$, the necessary condition for Stage 1 is no longer applicable since Stage 1 orders directly from Stage 3 with lead time $l_2 + l_1$ and we have to revise R_1^* accordingly.

Similarly, for each Stage $j > 2$, given R_1^*,....,R_{j-1}^*, the same results can be obtained. Consequently, R_j^* provides a unique minimum optimal solution. □

According to Theorem 2, under an optimal solution, the marginal cost at an echelon is equal to the inventory holding cost at the upstream stage. This is intuitive. Consider a unit of stock at Stage j at a review point of Stage $j-1$ and the decision between retaining it at Stage j or shipping it to the downstream Echelon $j-1$. The marginal cost of the first option is the holding cost h_j. Apparently, this unit of stock should be shipped to the downstream Echelon $j-1$ if the marginal cost at Echelon $j-1$ is lower than h_j. Following this intuition, an optimal inventory position (or order-up-to level) at the downstream Stage $j-1$ is reached when the two marginal costs are equal. This finding provides a simple characterization of an optimal ordering policy for a serial inventory system. Under this decision rule, an optimal base stock level at a stage is determined independently of the base stock levels and reorder intervals at the upstream stages. This independence condition leads to the bottom-up solution procedure in Theorem 2.

5. Effect of cost accounting methods

We examine the effects of applying discrete-time cost accounting methods when inventory holding and backordering costs actually accrue continuously in time in this section. We address three issues: (1) how significant is the effect of end-of-period cost accounting for single-stage systems, (2) how the inefficiency of end-of-period cost accounting carries over to multiple-stage systems, and (3) whether extending end-of-period cost accounting to discrete-time cost accounting that evaluates inventory related costs at multiple discrete points in a reorder interval provides any advantage as compared to the continuous-time solution we develop in this study.

Regardless of the cost accounting method, the same demand and continuous-time cost parameters are used. For a given replenishment schedule $\mathbf{T} = \{T_1, \ldots, T_N\}$, we let $\mathbf{R}^*(\mathbf{T}) = (R_1^*(\mathbf{T}), \ldots R_N^*(\mathbf{T}))$ denote the optimal ordering policy $\mathbf{R}$ that minimizes the system cost $C_N(R_N, T_N| \cdot)$. The minimum system cost is then given by $C_N(R_N^*, T_N|\cdot)$. We use $C_N(R_N^*(\mathbf{T})), T_N|\cdot)$ as the benchmark to evaluate the cost inefficiency of a discrete-time cost accounting method.

5.1. Effect of end-of-period cost accounting for single-Stage systems

We start with single-stage systems. There are two reasons. First, a single-stage system is equivalent to the newsvendor model that is widely used in the literature. Second, the effect of cost accounting on Stage 1 is of primary importance in analyzing the effect of cost accounting on multiple-stage systems since cost accounting method affects directly only the cost evaluation at Stage 1. We consider end-of-period cost accounting. In contrast to Rudi et al. (2009), we first illustrate numerically the significance of the excess stock and cost inefficiency caused by end-of-period cost accounting and subsequently prove that they are bounded below by the ratio of the inventory holding cost to the backordering cost asymptotically as the reorder interval increases.

We use the same notation above with a superscript "e" to differentiate end-of-period cost accounting. According to Lemma 1, when end-of-period cost accounting is used, the expected inventory holding and backordering cost at Stage 1 in a reorder interval is evaluated by $\pi_1^e(R_1, T_1) = u(R_1, l_1 + T_1)T_1$ and hence the average cost per unit time is equal to $C_1^e(R_1, T_1) = u(R_1, l_1 + T_1)$. Let $R_1^e(T_1)$ denote the base stock R_1 that minimizes $C_1^e(R_1, T_1)$ for a given T_1. When this base stock level is implemented, the actual average cost per unit time is equal to $C_1(R_1^e(T_1), T_1)$. We assess the stock increase and cost inefficiency caused by end-of-period cost accounting by $\Delta_S^1 = R_1^e(T_1)/R_1^*(T_1) - 1$ and $\Delta_C^1 = C_1(R_1^e(T_1), T_1)/C_1(R_1^*(T_1), T_1) - 1$ respectively. It is noted that the cost inefficiency Δ_C^1 is completely determined by the optimal base stock levels $R_1^e(T_1)$ and $R_1^*(T_1)$.

We conduct a numerical experiment with λ, T_1, $l_1 \in \{0.1, 2.1, 4.1, 6.1\}$ and $h_1/b \in \{0.01, 0.1, 1\}$ with $h_1 = 1$. We choose different values for these system parameters to investigate their impact on the effect of different cost accounting methods. We consider the ratio h_1/b in the investigation because we found that the effect of cost accounting method depends on the ratio h_1/b rather than the individual values of h_1 and b (see Theorem 3 and its proof in Appendix C). The experiment contains 192 different cases. In view of the literature, where most studies on stochastic periodic-review inventory control policies assumed Poisson demand, we also adopted this demand model to facilitate comparison. Our analysis holds under the discrete Possion demand model when aggregation over demand are modified from integration to summation in the cost evaluations. The conditions of optimality are modified accordingly.

Table 1 summarizes the statistics for the 192 cases in the numerical study. Overall, system cost increases by 20.29% on average with a standard deviation of 24.49%, ranging from 0.00% to 95.92%. The numerical study provides strong evidence against the use of end-of-period cost accounting. In addition, Fig. 3 shows that the cost inefficiency increases with the demand rate λ, the reorder interval T_1 and the ratio h_1/b but decreases with the leadtime l_1. A scrutiny of the individual cases reveals further that the cost inefficiency increases with λT_1 starting from zero for small λT_1. Since the variance of the demand in the reorder interval is λT_1 under the Poisson demand model, this means that the effect of end-of-period of cost accounting depends on the demand variation in the reorder interval. Excess stock exhibits a similar pattern.

This is reasonable. Under fixed-interval ordering, stock is ordered at a reorder point to meet the demand in the leadtime l_1 and the reorder interval T_1. When continuous-time cost accounting is used, the optimal base stock level $R_1^*(T_1)$ is determined based on the average inventory level in the reorder interval that is evaluated continuously in time. In contrast, when end-of-period cost accounting is used, the optimal base stock level $R_1^e(T_1)$ is determined based on the inventory level only at the end of the reorder interval. Because inventory on hand decreases and backorder increases over time, the inventory level at the end of a reorder interval underestimates the average inventory on hand and overestimates the average backorder in the period. As a result, end-of-period cost accounting entails an excess stock, i.e. $R_1^e(T_1) \geq R_1^*(T_1)$. Obviously, the difference between the two base stock levels depends on the demand deviation in the reorder interval. More specifically, the difference is zero if demand does not change within the reorder in-

Table 1
Results for single-stage system.

Cases		Deviation(%)	Mean	Std	Min	Max	Cases		Deviation(%)	Mean	Std	Min	Max
All		Δ_C^1	20.29	24.49	0	95.92	λ	0.1	Δ_C^1	2.12	6.50	0	36.05
		Δ_S^1	20.30	27.28	0	125.00			Δ_S^1	9.42	26.68	0	100.00
h_1/b	1	Δ_C^1	36.26	34.96	0	95.92		2.1	Δ_C^1	22.18	21.64	0	84.58
		Δ_S^1	31.79	37.57	0	125.00			Δ_S^1	25.36	28.80	0	125.00
	0.1	Δ_C^1	15.57	13.25	0	36.05		4.1	Δ_C^1	27.16	25.73	0	86.27
		Δ_S^1	17.97	21.10	0	100.00			Δ_S^1	23.92	27.84	0	125.00
	0.01	Δ_C^1	9.77	8.27	0	21.67		6.1	Δ_C^1	29.06	28.10	0	95.92
		Δ_S^1	11.68	14.91	0	100.00			Δ_S^1	22.12	23.42	0	100.00
T_1	0.1	Δ_C^1	0.34	1.77	0	12.04	l_1	0.1	Δ_C^1	26.04	29.26	0	95.92
		Δ_S^1	1.77	7.58	0	50.00			Δ_S^1	38.21	41.53	0	125.00
	2.1	Δ_C^1	19.03	17.82	0	76.99		2.1	Δ_C^1	20.93	24.87	0	89.45
		Δ_1	22.06	28.07	0	125.00			Δ_S^1	19.92	22.83	0	100.00
	4.1	Δ_C^1	28.62	25.65	0	84.58		4.1	Δ_C^1	18.27	22.43	0	83.44
		Δ_S^1	26.31	29.11	0	125.00			Δ_S^1	12.77	13.13	0	44.83
	6.1	Δ_C^1	33.18	28.58	0	95.92		6.1	Δ_C^1	15.99	20.18	0	77.86
		Δ_S^1	31.02	28.70	0	116.67			Δ_S^1	10.52	11.60	0	50.00

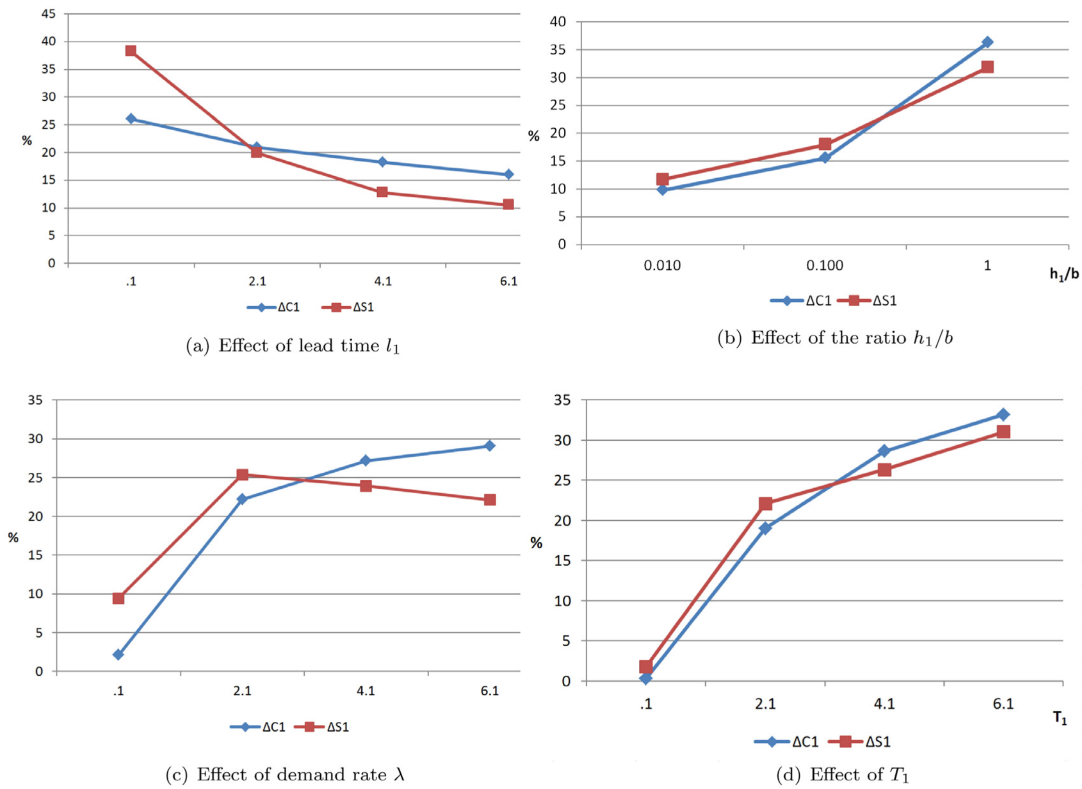

Fig. 3. Effect of system parameters on the inefficiency by end-of-period cost accounting.

terval and would become larger as the demand deviation in the reorder interval increases. This is what we observed in the numerical study. In addition, because the demand variation in the reorder interval T_1 relative to the demand variation in the whole period $l_1 + T_1$ decreases as the leadtime l_1 increases, the excess stock would decrease with l_1. Finally, the excess stock would be higher for a higher inventory holding cost h_1 and/or a lower backordering cost b. It is easy to verify that both Δ_S^1 and Δ_c^1 depend on the ratio h_1/b rather than their individual values. Theorem 3 proves that both the excess stock and cost inefficiency caused by end-of-period cost accounting are bounded below by the ratio h_1/b asymptotically as the reorder interval T_1 increases when demand follows the Brownian motion model with Normal density.

Theorem 3. *Let demand follow the Brownian motion model with Normal density (Rao, 2003):*

$$g(x,t) = \frac{1}{\sigma\sqrt{2\pi t}} exp[-\frac{(x-\lambda t)^2}{2\sigma^2 t}]. \tag{12}$$

Then Δ_S^1 and Δ_C^1 are bounded below by the ratio h_1/b asymptotically as the reorder interval T_1 increases. i.e., $lim_{T_1\to\infty}\Delta_S^1 \geq h_1/b$ and $lim_{T_1\to\infty}\Delta_C^1 \geq h_1/b$.

Proof. We place the proof of the theorem in Appendix C. □

5.2. Effect of end-of-period cost accounting for multiple-Stage systems

Our second task is to investigate how the cost inefficiency of end-of-period cost accounting at Stage 1 carries over to multiple-stage serial systems.

For this purpose, we conducted a numerical study that extended single-stage systems to two-stage systems and subsequently two-stage systems to three-stage systems. Due to computational considerations, we selected 16 single-stage systems for $\lambda \in \{2.1, 4.1\}$, T_1, $l_1 \in \{0.1, 2.1\}$ and $b \in \{1, 10\}$ with $h_1 = 1$ from the experiment above. First, we extended the 16 cases to two-stage serial systems with $n_2 \in \{1, 3\}$, $l_2 \in \{0.1, 2.1\}$ and $h_1 - h_2 \in \{0.1, 0.3\}$ and evaluated the excess stock and system cost increases. This numerical study contains 128 different cases. Subsequently we extended the 128 two-stage serial systems to three-stage serial systems with $n_3 \in \{1, 3\}$, $l_3 \in \{0.1, 2.1\}$ and $h_2 - h_3 \in \{0.1, 0.3\}$. This numerical study contains 1024 different cases.

We extended the notation and evaluation above from single-stage systems to multiple-stage systems. Specifically, given $\pi_1^e(R_1, T_1)$, we evaluate $\pi_j^e(R_j, T_j|\cdot)$ according to Theorem 1 for $j = 2, \ldots, N$ and $C_N^e(R_N, T_N|\cdot) = \pi_N^e(R_N, T_N|\cdot)/T_N$. Similar to the above, we let $\mathbf{R}^e(\mathbf{T}) = (R_1^e(\mathbf{T}), \ldots R_N^e(\mathbf{T}))$ denote the ordering policy $\mathbf{R}$ that minimizes $C_N^e(R_N, T_N|\cdot)$ for a given replenishment schedule $\mathbf{T}$. When this ordering policy is implemented, the actual system cost is $C_N(R_N^e(\mathbf{T}), T_N|\cdot)$. We use $\Delta_S^N = \sum_{j=1}^N R_j^e(\mathbf{T}) / \sum_{j=1}^N R_j^*(\mathbf{T}) - 1$ and $\Delta_C^N = C_N(R_N^e(\mathbf{T}), T_N|\cdot)/C_N(R_N^*(\mathbf{T}), T_N|\cdot) - 1$ to evaluate the system excess stock and cost inefficiency caused by end-of-period cost accounting for an N-stage serial system respectively.

Table 2 summarizes the statistics from the numerical experiment.

Table 2
Cost inefficiency and excess stock for systems with different number of stages.

No. of Stages	Deviation	Mean	Std	Min	Max
$N = 1$	$\Delta_C^1(\%)$	19.85	22.54	0	76.99
	$\Delta_S^1(\%)$	30.92	38.46	0	125.00
$N = 2$	$\Delta_C^2(\%)$	11.08	15.02	0	69.98
	$\Delta_S^2(\%)$	17.21	25.06	0	150.00
$N = 3$	$\Delta_C^3(\%)$	6.64	9.86	0	78.30
	$\Delta_S^3(\%)$	9.62	15.03	0	150.00

Table 3
Base planning interval and approximation accuracy.

Cases		$T_1 = 2.1, b = 1$		$T_1 = 2.1, b = 10$		$T_1 = 4.1, b = 1$		$T_1 = 4.1, b = 10$	
		$\Delta_C(\%)$	Time(s)	$\Delta_C(\%)$	Time(s)	$\Delta_C(\%)$	Time(s)	$\Delta_C(\%)$	Time(s)
Continuous		1	0.049	1	0.058	1	0.017	1	0.043
m	1	17.887	0.053	12.100	0.052	41.046	0.022	22.321	0.052
	2	6.318	0.037	4.463	0.056	13.899	0.023	7.884	0.048
	3	0.739	0.038	0.740	0.052	5.325	0.026	2.642	0.053
	4	0.739	0.035	0.118	0.061	0.694	0.023	2.642	0.056
	5	0.314	0.040	0.118	0.063	0.694	0.023	2.642	0.059
	6	0.314	0.042	0.118	0.064	0.694	0.025	0.650	0.048
	7	0.314	0.038	0.118	0.064	0.614	0.029	0.650	0.048
	8	0.314	0.047	0.118	0.065	0.614	0.031	0.000	0.078
	9	0.314	0.048	0.118	0.071	0.614	0.032	0.000	0.075
	10	0.314	0.047	0.118	0.073	0.614	0.035	0.000	0.071

According to Table 2, on average, system stock and system cost caused by end-of-period cost accounting increase respectively by 30.92% and 19.85% for the 16 single-stage systems; 17.21% and 11.08% for the 128 two-stage systems; and 9.62% and 6.62% for the 1024 three-stage systems. This shows that both excess stock and cost inefficiency decrease as the supply chain becomes longer. This is because cost accounting method affects only the cost evaluation at Stage 1 and therefore the carry-over effect on upperstream stages will diminish as the number of stage increases. Nevertheless, a supply chain in reality has typically only a few stages. According to our numerical studies, the effect of end-of-period cost accounting is still quite significant for three-stage serial inventory systems.

5.3. Approximation by discrete-time cost accounting

Finally, we note that, to alleviate the problem of end-of-period cost accounting, previous studies have used discrete-time cost accounting that evaluates inventory related costs at multiple discrete time points in each reorder interval at Stage 1 (Chao and Zhou, 2009; Shang and Zhou, 2010). These studies assumed that a base planning interval (typically normalized to length 1) has been chosen exogenously and all reorder intervals and lead times are integer multiples of the base planning interval. Inventory related costs are evaluated based on the inventory levels at the end of each base planning interval rather than only at the end of a reorder interval at Stage 1.

This discrete-time cost accounting method provides an approximation to the continuous-time solution we develop in this study. According to our analysis above, this approximation improves end-of-period cost accounting and can achieve any level of accuracy by reducing the length of the base planning interval. In particular, this approximation can be 100% accurate if the demand variation in the base planning interval is sufficiently small. Consequently, in comparison to the continuous-time solution, the applicability of discrete-time cost accounting depends on whether an appropriate base planning period can be easily identified and the computational requirement for the identification.

We make two observations from our extensive numerical studies.

First, as compared to continuous-time cost accounting, end-of-period cost accounting requires less computation but its computational advantage diminishes as the number of stages increases. For the numerical experiment in Section 5.2, the average computational times for the optimal solutions under end-of-period cost accounting and continuous-time cost accounting are 0.0029 and

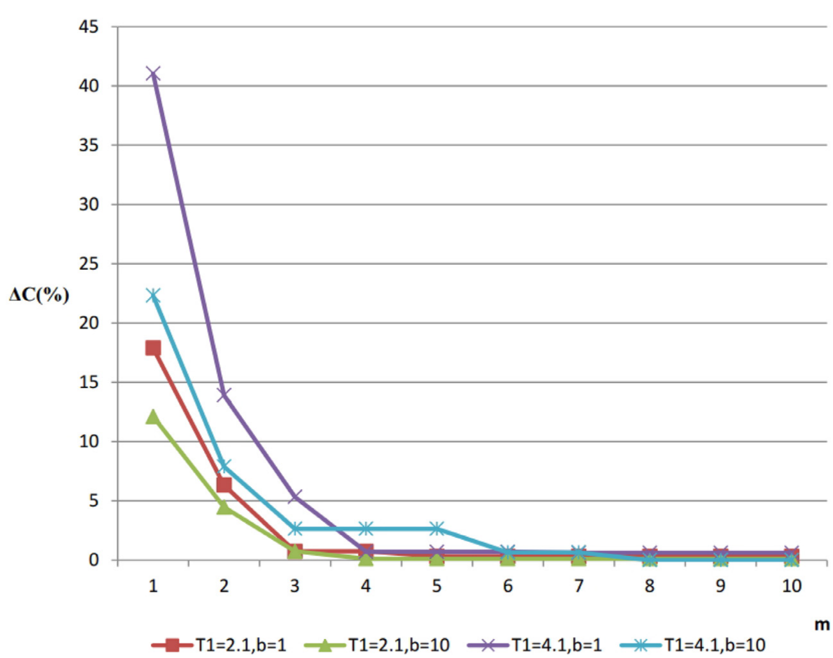

Fig. 4. Effect of base planning interval on approximation accuracy.

0.0106 seconds respectively for the 16 single-stage cases; 0.0193 and 0.0179s respectively for the 128 two-stage cases and 4.0984 and 4.7559s respectively for the 1024 three-stage cases.

Second, we consider discrete-time cost accounting and assess the impact of the base planning interval on cost deviation and the required computational time. We divide the reorder interval T_1 at Stage 1 into m equally spaced base planning intervals of length T_1/m, where m is a positive integer. According to Lemma 1, the inventory related cost at Stage 1 in a reorder interval is evaluated by $\pi_1(R_1, T_1) = \sum_{i=1}^{m} u(R_1, l_1 + iT_1/m)T_1/m$. Given $\pi_1(R_1, T_1)$, the rest of the analyses is the same to the case for end-of-period cost accounting above. Apparently, both approximation accuracy and computational requirement increase with m.

Our numerical experiment shows that the length of the base planning interval to achieve a 2% accuracy is case specific and the required computational time to achieve an approximation accuracy of 2% is in general comparative to that for the optimal solution under continuous-time cost accounting. For illustration purpose, we present the results in Table 3 for 4 three-stage systems, where $\lambda = 2.1,\ l_1 = l_2 = l_3 = 2.1,\ T_1 \in \{2.1, 4.1\},\ b \in \{1, 10\},\ n_2 = n_3 = 1,\ h_1 = 1,\ h_2 = 0.7$ and $h_3 = 0.4$. The computational time for the optimal solution under continuous-time cost accounting is presented in the first row. The cost inefficiency and computational time for the optimal solution under discrete-time cost accounting are presented below for $m = 1, 2, \ldots, 6$. As demonstrated in Fig. 4, the m that achieves a cost inefficiency of less than 2% is 3, 3, 4 and 6 for the four cases respectively. The computational time to achieve this accuracy is slightly smaller in the first two cases but slightly greater in the last two cases than the computational time for the respective optimal solution under continuous-time cost accounting. In general, our numerical studies show that discrete-time cost accounting does not provide a modeling and computational advantage as compared to continuous-time cost accounting.

6. Concluding remarks

This paper considers serial inventory control systems where inventory holding and backordering costs accrue continuously in time and investigates the impact of cost accounting methods on system stock and cost. We assume that the system adopts fixed-interval ordering and provide a bottom-up recursive approach to evaluate system cost and optimize the inventory control policy. We also prove that a unique optimal ordering policy is determined by simply equating the marginal echelon inventory related cost to the inventory holding cost at the upstream stage. Subsequently, we evaluate the effect of using discrete-time cost accounting methods to approximate the inventory control system numerically.

Our analysis shows that the adoption of end-of-period cost accounting method in a serial inventory system leads to excess stock and higher system cost as compared to the optimal solution under continuous-time cost accounting. Because end-of-period cost accounting ignores the demand variation in the reorder interval at the most downstream stage (i.e. Stage 1), we found that the cost inefficiency of end-of-period cost accounting increases with this demand variation. Moreover, since cost accounting method affects only the cost evaluation at Stage 1, the effect of end-of-period cost accounting on system cost will diminish as the number of stages increases. Nevertheless, our numerical studies show that the cost inefficiency is still quite significant for a typical serial inventory system. Our findings also show that, given the exact continuous-time cost accounting solution that is developed in the current study, discrete-time cost accounting generally does not provide any modelling and/or computational advantage. Given the wide adoption of end-of-period cost accounting, we believe that our study provides important insights for practitioners.

Future research may generalize the current study in two possible directions. First, since we assume that the system replenishment schedule is exogenously determined, a meaningful extension is to relax this assumption and investigate the impact of different cost accounting methods on the reorder intervals and base-stock levels. One conjecture is that end-of-period cost acconnting may entail a shorter reorder interval at Stage 1 so as to reduce the demand variation in the reorder interval and hence reduce the cost inefficiency caused by this demand variation. Another possible direction is to extend the current study to multi-echelon inventory systems with multiple retailers such as one-warehouse multi-retailer systems and assembly systems. We conjecture that the observations we make in this study hold for other more general multi-echelon systems. Nevertheless, the evaluation and optimization of such systems will be a challenge.

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Appendix A. Derivation of (7)

Changing the integration in (6) according to $\int_{R_j}^{\infty} = \int_0^{\infty} - \int_0^{R_j}$, we obtain

$$\sum_{q=0}^{n_j-1}\sum_{m=0}^{q} c_q^m(\cdot) = \sum_{q=0}^{n_j-1}\int_0^{\infty} \pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)g(y, \tau_{jq})dy - \sum_{q=0}^{n_j-1}\int_0^{R_j} \pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)g(y, \tau_{jq})dy). \tag{A.1}$$

Suppose that Stage $j-1$ orders directly from Stage $j+1$ at time 0, raising the IP at Echelon $j-1$ to S_j. Leadtime is $l_{j-1} + l_j$ and no other order will be made before time T_j. By definition, the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + l_j, l_{j-1} + l_j + T_j)$ is given by $\tilde{\pi}_{j-1}(R_j + R_{j-1}, T_j|\cdot)$ in this case. Moreover, given $Y_{jq} = y$, the IP at Echelon $j-1$ is equal to $S_j - y$ at time τ_{jq} and the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + \tau_{jq}, l_{j-1} + \tau_{jq+1})$ is then given by $\pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)$. Given this conditional cost evaluation, the expected cost at Echelon $j-1$ in the interval $[l_{j-1} + \tau_{jq}, l_{j-1} + \tau_{jq+1})$ is equal to $\int_0^{\infty} \pi_{j-1}(R_j + R_{j-1} - y, T_{j-1}|\cdot)g(y, \tau_{jq})dy$ and the sum of this expected cost over $q = 0, \ldots, n_j - 1$ is the expected cost in the interval $[l_{j-1} + l_j, l_{j-1} + l_j + T_j)$, which is $\tilde{\pi}_{j-1}(R_j + R_{j-1}, T_j|\cdot)$. This leads to (7).

Appendix B. Proof of Lemma 2

We differentiate $C_N(R_N, T_N|\cdot)$ with respect to R_j for $j = 1, \ldots, N$ and obtain

$$\frac{\partial C_N(R_N|\cdot)}{\partial R_N} = h_N \Phi^N(R_N) + \int_{R_N}^{\infty} \frac{\partial C_{N-1}(R_N + R_{N-1} - y|\cdot)}{\partial R_N}\phi^N(y)dy \text{ and} \tag{B.1}$$

$$\frac{\partial C_N(R_N|\cdot)}{\partial R_j} = \frac{\partial C_{N-1}(R_{N-1}|\cdot)}{\partial R_j}\Phi^N(R_N) + \int_{R_N}^{\infty} \frac{\partial C_{N-1}(R_N + R_{N-1} - y|\cdot)}{\partial R_j}\phi^N(y)dy \text{ for } j < N \tag{B.2}$$

For brevity, we keep only the base-stock level for the most upstream stage in the relevant functions.

Using (B.1) and (B.2), we obtain

$$\frac{\partial C_N(R_N|\cdot)}{\partial R_{N-1}} - \frac{\partial C_N(R_N|\cdot)}{\partial R_N} = [\frac{\partial C_{N-1}(R_{N-1}|\cdot)}{\partial R_{N-1}} - h_N]\Phi^N(R_N) \tag{B.3}$$

since $\partial C_{N-1}(R_N + R_{N-1} - y|\cdot)/\partial R_N = \partial C_{N-1}(R_N + R_{N-1} - y|\cdot)/\partial R_{N-1}$ for any $y \geq 0$ and

$$\frac{\partial C_N(R_N|\cdot)}{\partial R_j} - \frac{\partial C_N(R_N|\cdot)}{\partial R_{j+1}} = [\frac{\partial C_{N-1}(R_{N-1}|\cdot)}{\partial R_j} - \frac{\partial C_{N-1}(R_{N-1}|\cdot)}{\partial R_{j+1}}]\Phi^N(R_N) + \int_{R_N}^{\infty}[\frac{\partial C_{N-1}(R_N + R_{N-1} - y|\cdot)}{\partial R_j} - \frac{\partial C_{N-1}(R_N + R_{N-1} - y|\cdot)}{\partial R_{j+1}}]\phi^N(y)dy \text{ for } j < N. \tag{B.4}$$

We prove the relationship in Lemma 2 using induction with respect to N.

First, for $N = 2$, the relationship holds according to (B.3).

Second, for any $N > 2$, we assume that the relationship holds for $C_{N-1}(R_{N-1}|\cdot)$ or specifically

$$\frac{\partial C_{N-1}(R_{N-1}|\cdot)}{\partial R_j} - \frac{\partial C_{N-1}(R_{N-1}|\cdot)}{\partial R_{j+1}} = [\frac{\partial C_j(R_j|\cdot)}{\partial R_j} - h_{j+1}]P_j^{N-1}(R_{N-1}|\cdot) \text{ for } j < N-1 \tag{B.5}$$

where $P_j^{N-1}(R_{N-1}|\cdot)$ is a function of $R_1, \ldots, R_{N-1}$.

Finally, for $C_N(R_N| \cdot)$, the relationship holds for $j = N-1$ according to (B.3) and can be proved for $j < N-1$ by applying (B.5) in (B.4):

$$\frac{\partial C_N(R_N|\cdot)}{\partial R_j} - \frac{\partial C_N(R_N|\cdot)}{\partial R_{j+1}} = [\frac{\partial C_j(R_j|\cdot)}{\partial R_j} - h_{j+1}]P_j^{N-1}(R_{N-1}|\cdot)\Phi^N(R_N) + \int_{R_N}^{\infty}[\frac{\partial C_j(R_j|\cdot)}{\partial R_j} - h_{j+1}]P_j^{N-1}(R_N + R_{N-1} - y|\cdot)\phi^N(y)dy = [\frac{\partial C_j(R_j|\cdot)}{\partial R_j} - h_{j+1}]P_j^N(R_N|\cdot) \tag{B.6}$$

where $P_j^N(R_N|\cdot) = P_j^{N-1}(R_{N-1}|\cdot)\Phi^N(R_N) + \int_{R_N}^{\infty} P_j^{N-1}(R_N + R_{N-1} - y|\cdot)\phi^N(y)dy$.

Appendix C. Proof of Theorem 3

The average cost under end-of-period cost accounting is given by $C_1^e(R_1, T_1) = u(R_1, l_1 + T_1)$ and $R_1^e(T_1)$ is then determined by $\partial C_1^e(R_1, T_1)/\partial R_1 = 0$ or $G(R_1^e(T_1), l_1 + T_1) = \omega$, where $\omega = b/(b + h_1)$. We use the Normal standardization $G(R_1^e(T_1), l_1 + T_1) = \int_{-\infty}^{[R_1^e(T_1) - \lambda(l_1+T_1)]/(\sigma\sqrt{l_1+T_1})} \phi(y)dy$, where $\phi(y)$ is the probability density function for the standard Normal distribution, in differentiating $G(R_1^e(T_1), l_1 + T_1) = \omega$ w.r.t T_1 and obtain

$$R_1^e(T_1)' = \frac{\lambda}{2} + \frac{R_1^e(T_1)}{2(l_1 + T_1)}. \tag{C.1}$$

Th solution of the differential equation yields $R_1^e(T_1) = \lambda(l_1 + T_1) + c\sqrt{l_1 + T_1}$ and therefore $R_1^e(T_1)' = \lambda + c/(2\sqrt{l_1 + T_1})$, where c is a constant. This leads to $lim_{T_1\to\infty} R_1^e(T_1)' = \lambda$.

On the other hand, the average cost under continuous-time cost accounting is given by $C_1(R_1, T_1) = \int_{l_1}^{l_1+T_1} u(R_1, t)dt$ and therefore $R_1^*(T_1)$ is determined by $\partial C_1(R_1, T_1)/\partial R_1 = 0$ or $\int_{l_1}^{l_1+T_1} G(R_1^*(T_1), t)dt = \omega T_1$. Similar to the above, the differentiation of this equation w.r.t T_1 yields

$$R_1^*(T_1)' = \lambda\frac{\omega - G(R_1^*(T_1), l_1 + T_1)}{G(R_1^*(T_1), l_1) - G(R_1^*(T_1), l_1 + T_1)}. \tag{C.2}$$

Using the L'Hospital's Rule, we obtain

$$lim_{T_1\to\infty}\frac{R_1^e(T_1)}{R_1^*(T_1)} = \frac{lim_{T_1\to\infty}R_1^e(T_1)'}{lim_{T_1\to\infty}R_1^*(T_1)'} = \frac{1-\delta}{\omega - \delta} \geq \frac{1}{\omega} \tag{C.3}$$

since $lim_{T_1\to\infty} G(R_1^*(T_1)) = 1$, where $\delta = lim_{T_1\to\infty} G(R_1^*(T_1), l_1 + T_1)$.

This leads to $lim_{T_1\to\infty}\Delta_S^1 \geq 1/\omega - 1 = h_1/b$.

In addition, since $\omega = 1/(1 + h_1/b)$, we observe from the conditions of optimality above that $R_1^e(T_1)$ and $R_1^*(T_1)$, and hence Δ_1, depend on only the ratio h_1/b rather than the absolute values of h_1 and b.

We consider now the system cost. Let $\mu = C_1(R_1^e(T_1), T_1)/C_1(R_1^*(T_1), T_1)$. We first obtain $\mu = \int_{l_1}^{l_1+T_1}[(1 + h_1/b)\int_0^{R_1^e(T_1)} G(x, t)dx + \lambda t - R_1^e(T_1)]dt / \int_{l_1}^{l_1+T_1}[(1 + h_1/b)\int_0^{R_1^*(T_1)} G(x, t)dx + \lambda t - R_1^*(T_1)]dt$ and observe that μ and hence $\Delta_C^1 = \mu - 1$ depend on only the ratio h_1/b rather than the absolute values of h_1 and b. Next, we apply the L'Hospital's Rule again and obtain

$$lim_{T_1\to\infty}\mu = lim_{T_1\to\infty}\frac{u(R_1^e(T_1), l_1 + T_1) + R_1^e(T_1)'\int_{l_1}^{l_1+T_1} u_R(R^e(T_1), t)dt}{u(R_1^*(T_1), l_1 + T_1)} = lim_{T_1\to\infty}\frac{u_T(R^e(T_1), l_1 + T_1) + R_1^e(T_1)''\int_{l_1}^{l_1+T_1} u_R(R^e(T_1), t)dt + [R_1^e(T_1)']^2(b + h_1)\gamma(R_1^e(T_1), T_1)}{u_T(R_1^*(T_1), l_1 + T_1) + u_R(R_1^*(T_1), l_1 + T_1)R_1^*(T_1)'}. \tag{C.4}$$

where $u_R(R, t) = (b + h_1)G(R, t) - b$, $u_T(R, l_1 + T) = (b + h_1)\int_0^R G_T(x, l_1 + T)dx + b\lambda$, $G_T(x, l_1 + T) = \partial G(x, l_1 + T)/\partial T$, and $\gamma(R, T) = \int_{l_1}^{l_1+T} g(R, t)dt$.

We have used $\int_{l_1}^{l_1+T_1} u_R(R^*(T_1), t)dt = 0$ in Step 1 and $u_R(R_1^e(T_1), l_1 + T_1) = 0$ in Step 2.

To evaluate the right hand side of (C.4), we obtain the following. First, using the normal standardization above, we obtain $G_T(x, l_1 + T) = -\frac{\lambda g(x, l_1+T)}{\sigma\sqrt{l_1+T}} - \frac{[x - \lambda(l_1+T)]g(x, l_1+T)}{2\sigma(l_1+T)\sqrt{l_1+T}}$, $\int_0^R G_T(x, l_1 + T)dx = -\frac{\lambda}{\sigma\sqrt{l_1+T}}G(R, l_1 + T) + \frac{\sigma}{2\sqrt{l_1+T}}g(R, l_1 + T)$ and hence $lim_{T_1\to\infty} u_T(R, l_1 + T_1) = b\lambda$ for any $R \geq 0$. Second, following Rao (2003), we use $\gamma(R_1(T_1), T_1) \approx [G(R_1(T_1), l_1) - G(R_1(T_1), l_1 + T_1)]/\lambda$ and obtain $lim_{T_1\to\infty}\gamma(R_1^e(T_1), T_1) = (1 - \omega)/\lambda$ since $G(R_1^e(T_1), l_1 + T_1) = \omega$ and $lim_{T_1\to\infty} G(R_1^e(T_1), l_1) = 1$. Finally, using the solution of $R_1^e(T_1)$ above, we obtain $lim_{T_1\to\infty} R_1^e(T_1)'' = 0$. The substitution of these results as well as $lim_{T_1\to\infty} R_1^e(T_1)'$ and $lim_{T_1\to\infty} R_1^*(T_1)'$ as given above into the right hand side of (C.4) yields $lim_{T_1\to\infty}\mu = \{b\lambda + \lambda^2(b + h_1)[h_1/(b + h_1)]/\lambda\}/\{b\lambda + [(b + h_1)(1 - \delta) - b]\lambda[1 - (1 - \omega)/\delta]\} = 1/[\omega - (1 - \omega - \delta)^2/\delta]$ after simplification. It is easy to verify that $(1 - \omega - \delta)^2/\delta$ and hence $1/[\omega - (1 - \omega - \delta)^2/\delta]$ are increasing functions of δ and therefore $lim_{T_1\to\infty}\mu \geq 1/\omega$ since $1 - \omega < \delta < 1$. This proves $lim_{T_1\to\infty}\Delta_C^1 \geq h_1/b$.

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